

Clusters of bound particles in a quantum integrable many-body system and number theory

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Abstract

We construct clusters of bound particles for a quantum integrable derivative δ -function Bose gas in one dimension. It is found that clusters of bound particles can be constructed for this Bose gas for some special values of the coupling constant, by taking the quasi-momenta associated with the corresponding Bethe state to be equidistant points on a *single* circle in the complex momentum plane. Interestingly, there exists a connection between the above mentioned special values of the coupling constant and some fractions belonging to the Farey sequences in number theory. This connection leads to a classification of the clusters of bound particles for the derivative δ -function Bose gas and the determination of various properties of these clusters like their size and their stability under a variation of the coupling constant.

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1 Introduction

One-dimensional (1D) quantum integrable many-body systems with short range interactions have emerged as an active area of research [1]- [20], due to their effectiveness in describing recent experiments using strongly interacting ultracold atomic gases [21]- [27]. Indeed, many results of these experiments have been understood within the framework of the 1D quantum integrable Lieb-Liniger model or the δ -function Bose gas, which can be solved exactly through the Bethe ansatz.

In this context it may be recalled that, for a large class of quantum integrable systems, the method of coordinate Bethe ansatz directly yields exact eigenfunctions in the coordinate representation. One can study the asymptotic form of these eigenfunctions in the limit of infinite length of the system (i.e., when all x_i 's are allowed to take value in the range $-\infty < x_i < \infty$). If the probability density associated with an eigenfunction decays sufficiently fast when the relative distance between any two particle coordinates tends towards infinity (for a translationally invariant system), a bound state is formed. It is well known, for the case of the δ -function Bose gas with $N \geq 2$, that bound states exist for all negative values of the coupling constant [2, 3, 5-9]. The quasi-momenta associated with such a bound state are represented by equidistant points lying on a straight line or 'string' parallel to the imaginary axis in the complex momentum plane. For the case of the δ -function Bose gas with negative values of the coupling constant, one can also construct Bethe eigenfunctions corresponding to more complex structures like clusters of bound particles. The quasi-momenta corresponding to such clusters of bound particles are represented through discrete points lying on several 'strings', all of which are parallel to the imaginary axis in the complex momentum plane [3, 9, 12-15].

Similar to the case of the δ -function Bose gas mentioned above, there exists another exactly solvable and quantum integrable bosonic system with a Hamiltonian given by

$$\mathcal{H}_N = -\hbar^2 \sum_{j=1}^N \frac{\partial^2}{\partial x_j^2} + 2i\hbar^2\eta \sum_{l < m} \delta(x_l - x_m) \left(\frac{\partial}{\partial x_l} + \frac{\partial}{\partial x_m} \right), \quad (1.1)$$

where we have chosen $2m = 1$ and η is a real (nonzero) dimensionless coupling constant for this choice of mass [28-32]. The Hamiltonian (1.1) of this derivative δ -function Bose gas can be obtained by projecting that of an integrable derivative nonlinear Schrödinger (DNLS) quantum field model on the N -particle subspace. Classical and quantum versions of such DNLS field models have found applications in different areas of physics like circularly polarized nonlinear Alfvén waves in plasma, quantum properties of optical solitons in fibers, and in some chiral Tomonaga-Luttinger liquids obtained from the Chern-Simons model defined in two dimensions [33-39]. The scattering and bound states of the derivative δ -function Bose gas (1.1) have been studied extensively by using the methods of coordinate as well as algebraic Bethe ansatz [28-32, 40-42]. It turns out that for the cases $N = 2$ and $N = 3$, bound states of this model can be constructed for any value of η within its full range: $0 < |\eta| < \infty$. However, for any given value of $N \geq 4$, the derivative δ -function Bose gas allows bound states in only certain non-overlapping ranges of the coupling constant η (the union of these ranges yields a proper subset of

the full range of η), and such non-overlapping ranges of η can be determined by using the Farey sequences in number theory [40–42].

In analogy with the case of the δ -function Bose gas, one may think that clusters of bound particles can only be constructed for the case of derivative δ -function Bose gas by properly assigning the corresponding quasi-momenta on several concentric circles or circular ‘strings’ in the complex momentum plane. However, we have recently found that, for the Hamiltonian (1.1) with some special values of the coupling constant η , clusters of bound particles can be constructed in a much simpler way by assigning the corresponding quasi-momenta as equidistant points on a *single* circle in the complex momentum plane [43]. The purpose of the present article is mainly to review the key results of Ref. [43] and make some additional comment. The arrangement of this article is as follows. In Sec. 2, we first discuss the general form of Bethe eigenstates for the derivative δ -function Bose gas. Then we identify a sufficient condition for which such a Bethe eigenstate would represent clusters of bound particles. In Sec. 3, we classify all possible solutions of this sufficient condition and obtain different types of clusters of bound particles for the derivative δ -function Bose gas. In Sec. 4 we discuss various properties of such clusters of bound particles, such as the sizes of the clusters, their stability under the variation of the coupling constant. We end with some concluding remarks in Sec. 5.

2 Construction of clusters of bound particles

In the coordinate representation, the eigenvalue equation for the Hamiltonian (1.1) may be written as

$$\mathcal{H}_N \tau_N(x_1, x_2, \dots, x_N) = E \tau_N(x_1, x_2, \dots, x_N), \quad (2.1)$$

where $\tau_N(x_1, x_2, \dots, x_N)$ denotes a completely symmetric N -particle wave function. Since \mathcal{H}_N commutes with the total momentum operator given by

$$\mathcal{P}_N = -i\hbar \sum_{j=1}^N \frac{\partial}{\partial x_j}, \quad (2.2)$$

$\tau_N(x_1, x_2, \dots, x_N)$ can be chosen as a simultaneous eigenfunction of these two commuting operators. Note that \mathcal{H}_N remains invariant while \mathcal{P}_N changes sign if we change the sign of η and transform all the $x_i \rightarrow -x_i$ at the same time; such a transformation may be called as ‘parity transformation’. Due to the invariance of \mathcal{H}_N under this parity transformation, it is sufficient to study the eigenvalue problem (2.1) for one particular sign of η , say, $\eta > 0$. The eigenfunctions for $\eta < 0$ case can then be constructed from those for $\eta > 0$ case by simply changing $x_i \rightarrow -x_i$; this leaves all energy eigenvalues invariant but reverses the sign of the corresponding momentum eigenvalues.

For the purpose of solving the eigenvalue problem (2.1) through the coordinate Bethe ansatz, it is convenient to divide the coordinate space $R^N \equiv \{x_1, x_2, \dots, x_N\}$ into various N -dimensional sectors defined through inequalities like $x_{\omega(1)} < x_{\omega(2)} < \dots < x_{\omega(N)}$, where $\{\omega(1), \omega(2), \dots, \omega(N)\}$ represents a permutation of the integers $\{1, 2, \dots, N\}$.

Since the interaction part of the Hamiltonian (1.1) vanishes within each such sector, the resulting eigenfunction can be expressed as a superposition of free particle wave functions. The coefficients associated with these free particle wave functions can be computed by using the interaction part of the Hamiltonian (1.1), which is nontrivial only at the boundary of two adjacent sectors. In the region $x_1 < x_2 < \cdots < x_N$, such eigenfunctions can be written in the form [28, 30]

$$\tau_N(x_1, x_2, \cdots, x_N) = \frac{1}{\sqrt{N!}} \sum_{\omega} \left(\prod_{l < m} \frac{A(k_{\omega(m)}, k_{\omega(l)})}{A(k_m, k_l)} \right) \rho_{\omega(1), \omega(2), \cdots, \omega(N)}(x_1, x_2, \cdots, x_N) , \quad (2.3)$$

where

$$\rho_{\omega(1), \omega(2), \cdots, \omega(N)}(x_1, x_2, \cdots, x_N) = \exp \{ i(k_{\omega(1)}x_1 + \cdots + k_{\omega(N)}x_N) \} , \quad (2.4)$$

k_n 's are all distinct quasi-momenta, ω represents an element of permutation group for the integers $\{1, 2, \dots, N\}$ and \sum_{ω} implies summing over all such permutations. The coefficient $A(k_l, k_m)$ in Eq. (2.3) is obtained by solving the two-particle problem related to the derivative δ -function Bose gas and this coefficient is given by

$$A(k_l, k_m) = \frac{k_l - k_m + i\eta(k_l + k_m)}{k_l - k_m} . \quad (2.5)$$

The eigenvalues of the momentum (2.2) and Hamiltonian (1.1) operators, corresponding to the eigenfunctions $\tau_N(x_1, x_2, \cdots, x_N)$ of the form (2.3), are easily obtained as

$$\mathcal{P}_N \tau_N(x_1, x_2, \cdots, x_N) = \hbar \left(\sum_{j=1}^N k_j \right) \tau_N(x_1, x_2, \cdots, x_N) , \quad (2.6a)$$

$$\mathcal{H}_N \tau_N(x_1, x_2, \cdots, x_N) = \hbar^2 \left(\sum_{j=1}^N k_j^2 \right) \tau_N(x_1, x_2, \cdots, x_N) . \quad (2.6b)$$

Next, we shall discuss how Bethe states of the form in (2.3) lead to the bound states of the derivative δ -function Bose gas, by allowing k_j 's to take complex values in an appropriate way. As mentioned earlier, for a translationally invariant system, a wave function represents a localized bound state if the corresponding probability density decays sufficiently fast when any of the relative coordinates measuring the distance between a pair of particles tends towards infinity. To obtain the condition for which the Bethe state (2.3) would represent such a localized bound state, let us first consider the following wave function in the region $x_1 < x_2 < \cdots < x_N$:

$$\rho_{1,2,\dots,N}(x_1, x_2, \cdots, x_N) = \exp \left(i \sum_{j=1}^N k_j x_j \right) , \quad (2.7)$$

where k_j 's in general are complex valued wave numbers. Since the corresponding momentum eigenvalue given by $\hbar \sum_{j=1}^N k_j$ must be a real quantity, one obtains the condition

$$\sum_{j=1}^N q_j = 0 , \quad (2.8)$$

where q_j denotes the imaginary part of k_j . By using (2.8), the probability density for the wave function $\rho_{1,2,\dots,N}(x_1, x_2, \dots, x_N)$ in (2.7) can be expressed as

$$|\rho_{1,2,\dots,N}(x_1, x_2, \dots, x_N)|^2 = \exp \left\{ 2 \sum_{r=1}^{N-1} \left(\sum_{j=1}^r q_j \right) y_r \right\}, \quad (2.9)$$

where the y_r 's are the $N-1$ relative coordinates: $y_r \equiv x_{r+1} - x_r$. Hence, the probability density in (2.9) decays exponentially in the limit $y_r \rightarrow \infty$ for one or more values of r , provided that all the following conditions are satisfied:

$$q_1 < 0, \quad q_1 + q_2 < 0, \quad \dots, \quad \sum_{j=1}^{N-1} q_j < 0. \quad (2.10)$$

It should be observed that the wave function (2.7) is obtained by taking ω as the identity permutation in (2.4). However, the Bethe state (2.3) also contains terms like (2.4) with ω representing all possible nontrivial permutations. The conditions which ensure the decay of such a term, associated with any nontrivial permutation ω , are evidently given by

$$q_{\omega(1)} < 0, \quad q_{\omega(1)} + q_{\omega(2)} < 0, \quad \dots, \quad \sum_{j=1}^{N-1} q_{\omega(j)} < 0. \quad (2.11)$$

It is easy to check that above conditions, in general, contradict the conditions given in Eq. (2.10). To bypass this problem and ensure an overall decaying wave function (2.3), it is sufficient to assume that the coefficients of all terms $\rho_{\omega(1),\omega(2),\dots,\omega(N)}(x_1, x_2, \dots, x_N)$ with nontrivial permutations take the zero value. This leads to a set of relations given by

$$A(k_r, k_{r+1}) = 0, \quad \text{for } r \in \Omega_N, \quad (2.12)$$

where $\Omega_N \equiv \{1, 2, \dots, N-1\}$. Consequently, the simultaneous validity of the conditions (2.8), (2.10) and (2.12) ensures that the Bethe state $\tau_N(x_1, x_2, \dots, x_N)$ (2.3) would represent a bound state.

Let us now analyse the conditions (2.8), (2.10) and (2.12) for the case of the derivative δ -function Bose gas. Using the conditions (2.8) and (2.12) along with Eq. (2.5), one can easily derive an expression for all the quasi-momenta as

$$k_n = \chi e^{-i(N+1-2n)\phi}, \quad (2.13)$$

where χ is a real, non-zero parameter, and ϕ is related to the coupling constant η as

$$\phi = \tan^{-1}(\eta) \implies \eta = \tan \phi. \quad (2.14)$$

To obtain a unique value of ϕ from the above equation, it may be restricted to the fundamental region $-\frac{\pi}{2} < \phi (\neq 0) < \frac{\pi}{2}$. Furthermore, since we have seen that \mathcal{H}_N (1.1) remains invariant under the 'parity transformation', it is enough to study the corresponding eigenvalue problem only within the range $0 < \phi < \frac{\pi}{2}$. Next, we consider

the remaining conditions (2.10) for the existence of a localized bound state. Since summation over the imaginary parts of k_n 's in (2.13) yields

$$\sum_{j=1}^l q_j = -\chi \frac{\sin(l\phi)}{\sin \phi} \sin[(N-l)\phi], \quad (2.15)$$

Eq. (2.10) can be expressed as

$$\chi \frac{\sin(l\phi)}{\sin \phi} \sin[(N-l)\phi] > 0, \quad \text{for } l \in \Omega_N, \quad (2.16)$$

where Ω_N denotes the set of integers $\{1, 2, \dots, N-1\}$. Consequently, for any given values of ϕ and N , a bound state would exist when all the inequalities in Eq. (2.16) are simultaneously satisfied for some real non-zero value of χ . In our earlier works it was shown that, for any given value of $N \geq 4$, the derivative δ -function Bose gas allows bound states in only certain non-overlapping ranges of the coupling constant ϕ called 'bands' and the location of these bands can be determined exactly [40, 42].

We would now like to find the conditions for constructing clusters of bound particles in the case of the derivative δ -function Bose gas. To this end, we shall first discuss the concept of a 'clustered state' for any translationally invariant system, and then give a prescription for finding the Bethe states representing clusters of bound particles. Let us consider a system of N particles which are divided into some groups or clusters — with at least one group containing more than one particle. It is assumed that particles within the same group behave like the constituents of a bound state, but particles corresponding to different groups behave like the constituents of a scattering state. More precisely, a wave function corresponding to such an N -particle system satisfies the following two conditions. If the relative distance between any two particles belonging to the same group goes to infinity, the probability density corresponding to the N -particle wave function decays in the same way as a bound state. On the other hand, if the relative distance between any two particles belonging to different groups tends towards infinity (keeping the relative distances among all particles belonging to the same group unchanged), the probability density remains finite similar to a scattering state. If any wave function corresponding to a N -particle system satisfies these two conditions, we define it as a clustered state.

Next, let us discuss how the conditions for constructing a clustered state can be implemented for the case of the plane wave function (2.7). Since any eigenvalue of the momentum operator \mathcal{P}_N must be a real quantity, Eq. (2.8) is also obeyed for this case. To proceed further, let us choose a specific value of N given by $N = 4$. For this case, the probability density (2.9) may be explicitly written as

$$|\rho_{1,2,\dots,4}(x_1, x_2, \dots, x_4)|^2 = \exp \{2q_1 y_1 + 2(q_1 + q_2)y_2 + 2(q_1 + q_2 + q_3)y_3\}. \quad (2.17)$$

Suppose, the conditions (2.10) for a bound state formation are slightly modified for this case as

$$q_1 < 0, \quad q_1 + q_2 = 0, \quad q_1 + q_2 + q_3 < 0.$$

Taking into account this new condition, it is easy to see that when $y_1 = x_2 - x_1$ or $y_3 = x_4 - x_3$ tends towards infinity, the probability density in Eq. (2.17) still decays like a bound state. On the other hand, when $y_2 = x_3 - x_2$ tends towards infinity, the probability density in Eq. (2.17) remains finite. Hence, the clusters of particles given by $\{1, 2\}$ and $\{3, 4\}$ satisfy all the criteria of a clustered state. Generalizing this specific example in a straightforward way for any given values of N and ϕ , we replace some of the inequalities in Eq. (2.10) by equalities. In this way, we find out the conditions for obtaining a clustered state from the plane wave function (2.7) as

$$\sum_{i=1}^l q_i = 0, \quad \text{for } l \in \Omega_{N,\phi}, \quad (2.18a)$$

$$\sum_{i=1}^l q_i < 0, \quad \text{for } l \in (\Omega_N - \Omega_{N,\phi}), \quad (2.18b)$$

where $\Omega_{N,\phi}$ denotes any non-empty proper subset of Ω_N and $(\Omega_N - \Omega_{N,\phi})$ is the complementary set of $\Omega_{N,\phi}$.

Next, we try to find the simplest possible condition for which any Bethe state of the form (2.3) would represent clusters of bound particles. Let us assume that the quasi-momenta corresponding to this Bethe state satisfy the relations (2.12). As a result, the coefficients of all plane waves except (2.7) take the zero value within the Bethe state (2.3). Thus, due to the relations (2.12), the Bethe state (2.3) would reduce to the plane wave function (2.7). Next, we assume that the quasi-momenta corresponding to this Bethe state also satisfy the relations (2.8) and (2.18a,b). Hence, the plane wave function (2.7) represents a clustered state. Consequently, Eqs. (2.8), (2.12) and (2.18a,b) together yield a sufficient and simplest possible condition for which the Bethe state (2.3) would represent clusters of bound particles. Let us now analyse this condition for the case of the derivative δ -function Bose gas. Using Eqs. (2.8) and (2.12) along with the form of $A(k_l, k_m)$ given in (2.5) it is easy to see that, similar to the case of a localized bound state, the quasi-momenta associated with clusters of bound particles can be written in the form (2.13) and the imaginary parts of these quasi-momenta satisfy the relation (2.15). With the help of Eq. (2.15), we can recast Eqs. (2.18a,b) as

$$\chi \frac{\sin(l\phi)}{\sin\phi} \sin[(N-l)\phi] = 0, \quad \text{for } l \in \Omega_{N,\phi}, \quad (2.19a)$$

$$\chi \frac{\sin(l\phi)}{\sin\phi} \sin[(N-l)\phi] > 0, \quad \text{for } l \in (\Omega_N - \Omega_{N,\phi}). \quad (2.19b)$$

For any given values of N and ϕ , the above equations clearly give the simplest possible condition for which the Bethe state (2.3) would represent clusters of bound particles. Let us assume that these equations are satisfied for some values of ϕ and N , where $\Omega_{N,\phi}$ is given by

$$\Omega_{N,\phi} = \{l_1, l_2, \dots, l_p\}, \quad (2.20)$$

with $1 \leq p < N - 1$. Then from Eq. (2.9) it follows that the sets of particles given by $\{1, \dots, l_1\}, \{l_1 + 1, \dots, l_2\}, \dots, \{l_{p-1} + 1, \dots, l_p\}, \{l_p + 1, \dots, N\}$ represent $(p + 1)$

number of clusters of bound particles. Moreover, the numbers of particles present within each of these clusters, i.e., the size of the clusters, may be written in the form

$$\{\{l_1, l_2 - l_1, \dots, l_p - l_{p-1}, N - l_p\}\}. \quad (2.21)$$

Since the quasi-momenta associated with both bound states and clusters of bound particles are given by Eq. (2.13), the momentum and energy eigenvalues for clusters of bound particles can be derived in exactly the same way as has been done earlier [40] for the case of a bound state. Inserting the quasi-momenta given in Eq. (2.13) to Eqs. (2.6a,b), we obtain the momentum eigenvalue as

$$P = \hbar\chi \frac{\sin(N\phi)}{\sin\phi}, \quad (2.22)$$

and the energy eigenvalue as

$$E = \frac{\hbar^2\chi^2 \sin(2N\phi)}{\sin(2\phi)}. \quad (2.23)$$

Let us make a comment at this place. Using Eq. (2.22) we find that, for any given values of N and ϕ , the parameter χ is proportional to the total momentum P . Hence, due to Eqs. (2.9) and (2.15) it follows that, in the limit of large relative distances between particle coordinates, the probability density for a bound state wave function (or for a cluster within a clustered state) decays exponentially over a length which is inversely proportional to the total momentum P . This surprising result is a consequence of the fact that since we have set $2m = 1$ and the parameter η is dimensionless in the Hamiltonian (1.1), the only length scale appearing in an eigenstate of the Hamiltonian is \hbar/P . Hence the decay length must be proportional to \hbar/P .

3 Farey sequences and clusters of bound particles

For the purpose of constructing clusters of bound particles in the case of the derivative δ -function Bose gas, here our aim is to find out all possible solutions of Eqs. (2.19a,b). Some properties of the Farey sequences [44] in number theory will play a crucial role in our analysis. Due to the existence of the parity transformation, as mentioned in the earlier section, it is sufficient to concentrate on values of ϕ lying in the range $0 < \phi < \frac{\pi}{2}$. Within this range of ϕ , $\sin\phi > 0$ and hence the conditions (2.19a,b) for forming clusters of bound particles reduce to

$$\chi \sin(l\phi) \sin[(N-l)\phi] = 0, \quad \text{for } l \in \Omega_{N,\phi}, \quad (3.1a)$$

$$\chi \sin(l\phi) \sin[(N-l)\phi] > 0, \quad \text{for } l \in (\Omega_N - \Omega_{N,\phi}). \quad (3.1b)$$

It is easy to see that the condition (3.1a) would be satisfied if and only if ϕ/π can be expressed in the form

$$\frac{\phi}{\pi} = \frac{a}{b}, \quad (3.2)$$

where $\{a, b\}$ are relatively prime integers (i.e, the greatest common divisor of a and b is 1), taking values within the ranges

$$0 < a < \frac{b}{2}, \quad 2 < b \leq N - 1. \quad (3.3a, b)$$

Due to Eq. (3.3b) it is evident that, clusters of bound particles can exist only for $N \geq 4$. In the following, we shall establish a connection of the fractions ϕ/π , given by Eqs. (3.2) and (3.3a,b), with the elements of Farey sequences in number theory.

For a positive integer N , the Farey sequence is defined to be the set of all the fractions a/b in increasing order such that (i) $0 \leq a \leq b \leq N$, and (ii) $\{a, b\}$ are relatively prime integers [44]. The Farey sequences for the first few values of N are given by

$$\begin{aligned} F_1 : & \quad \frac{0}{1} \quad \frac{1}{1} \\ F_2 : & \quad \frac{0}{1} \quad \frac{1}{2} \quad \frac{1}{1} \\ F_3 : & \quad \frac{0}{1} \quad \frac{1}{3} \quad \frac{1}{2} \quad \frac{2}{3} \quad \frac{1}{1} \\ F_4 : & \quad \frac{0}{1} \quad \frac{1}{4} \quad \frac{1}{3} \quad \frac{1}{2} \quad \frac{2}{3} \quad \frac{3}{4} \quad \frac{1}{1} \\ F_5 : & \quad \frac{0}{1} \quad \frac{1}{5} \quad \frac{1}{4} \quad \frac{1}{3} \quad \frac{2}{5} \quad \frac{1}{2} \quad \frac{3}{5} \quad \frac{2}{3} \quad \frac{3}{4} \quad \frac{4}{5} \quad \frac{1}{1} \end{aligned} \quad (3.4)$$

These sequences enjoy several properties, of which we list the relevant ones below.

(i) Let $a/b, a'/b'$ are two fractions appearing in the Farey sequence F_N . Then $a/b < a'/b'$ ($a'/b' < a/b$) are two successive fractions in F_N , if and only if the following two conditions are satisfied:

$$a'b - ab' = 1 \quad (-1), \quad (3.5a)$$

$$b + b' > N. \quad (3.5b)$$

It then follows that both a and b' are relatively prime to a' and b .

(ii) For $N \geq 2$, if n/N is a fraction appearing somewhere in the sequence F_N (this implies that N and n are relatively prime according to the definition of F_N), then the fractions a_1/b_1 and a_2/b_2 appearing immediately to the left and to the right respectively of n/N satisfy

$$\begin{aligned} a_1, a_2 &\leq n, \quad \text{and} \quad a_1 + a_2 = n, \\ b_1, b_2 &< N, \quad \text{and} \quad b_1 + b_2 = N. \end{aligned} \quad (3.6)$$

To apply the above mentioned Farey sequence in the present context, let us define a subset of F_N as

$$F'_N = \left\{ \frac{a}{b} \mid \frac{a}{b} \in F_N, \quad \frac{0}{1} < \frac{a}{b} < \frac{1}{2} \right\}, \quad (3.7)$$

and a subset of F'_N as

$$F''_N = \left\{ \frac{n}{N} \mid \frac{n}{N} \in F'_N \right\}. \quad (3.8)$$

Using these definitions of various subsets of a Farey sequence, it is easy to show that

$$F'_N = F'_{N-1} \cup F''_N. \quad (3.9)$$

Furthermore, it is worth noting that, Eqs. (3.2) and (3.3a,b) can equivalently be expressed as

$$\frac{\phi}{\pi} \in F'_{N-1}. \quad (3.10)$$

Consequently, it follows that the condition (3.1a) for cluster formation is obeyed if and only if $\phi/\pi \in F'_{N-1}$. In this context it may be observed that, due to Eq. (3.9), all the elements of F'_{N-1} are also present in F'_N . By using such an embedding of F'_{N-1} into F'_N , we find that any fraction $a/b \in F'_{N-1}$ belongs to one of the four distinct classes, which are defined in the following:

- I. At least one of the fractions nearest to a/b (from either the left or the right side) in the sequence F'_N lies in the set F''_N . Then, from a property of the Farey sequences, it follows that $\{b, N\}$ are relatively prime integers in this case.
- II. None of the nearest fractions of a/b (from the left or right side) in the sequence F'_N lies in the set F''_N , and $\{b, N\}$ are relatively prime integers.
- III. N is divisible by b . Clearly, $\{b, N\}$ are not relatively prime integers in this case.
- IV. N is not divisible by b , and $\{b, N\}$ are not relatively prime integers.

To demonstrate the above mentioned classification through an example, let us choose $N = 6$. For this case, the sets F'_5 , F'_6 and F''_6 are given by

$$\begin{aligned} F'_5 : & \quad \frac{1}{5} \quad \frac{1}{4} \quad \frac{1}{3} \quad \frac{2}{5} \\ F'_6 : & \quad \frac{1}{6} \quad \frac{1}{5} \quad \frac{1}{4} \quad \frac{1}{3} \quad \frac{2}{5}; \quad F''_6 : \quad \frac{1}{6}. \end{aligned}$$

Using the embedding of F'_5 into F'_6 , it is easy to verify that each fraction in F'_5 falls under one of the four classes discussed above. More precisely, the fractions $1/5$, $1/4$, $1/3$ and $2/5$ belong to type I, type IV, type III and type II respectively. Returning back to the general case we note that, for any fraction $a/b \in F'_{N-1}$, $\{b, N\}$ are either relatively prime integers or not relatively prime integers. If $\{b, N\}$ are relatively prime integers, then it is obvious that a/b must be an element of either type I or type II. On the other hand, if $\{b, N\}$ are not relatively prime integers, then a/b must be an element of either type III or type IV. In this way, one can show that any fraction $a/b \in F'_{N-1}$ belongs to one of these four distinct classes.

Through a lengthy analysis which uses the properties (3.5a,b) and (3.6) of a Farey sequence, we have shown that fractions of types I and III belonging to F'_{N-1} satisfy the relation (3.1b), while type II and type IV fractions do not satisfy (3.1b) [43]. Hence, clusters of bound particles are formed for the case of derivative δ -function Bose gas, if and only if the corresponding coupling constant $\phi/\pi = a/b \in F'_{N-1}$ is a fraction of type I or type III.

4 Some properties of clusters of bound particles

In this section, we shall compute the number of clusters present within a Bethe state representing clusters of bound particles and the sizes of these clusters (i.e., number of bound particles present in each of these clusters). Subsequently, we shall analyse the behavior of these clusters of bound particles under small variations of the coupling constant.

4.1 Sizes of the clusters of bound particles

Let us first consider clusters of bound particles when $\phi/\pi = a/b$ is taken as any fraction of type I within the set F'_{N-1} . In section 2 we have seen that, to find the number and sizes of the clusters within a Bethe state, we have to determine the set $\Omega_{N,\phi}$ for which Eq. (3.1a) is satisfied. Since $\{N, b\}$ are relatively prime integers for any fraction of type I, we can express N as

$$N = pb + r, \quad (4.1)$$

where $1 \leq r \leq b - 1$. Hence, for the discrete variable l taking values within the set Ω_N , the zero points of the functions $\sin l\phi$ and $\sin(N - l)\phi$ are respectively given by the sets

$$S_1 \equiv \{b, 2b, \dots, pb\}, \quad S_2 \equiv \{N - b, N - 2b, \dots, N - pb\}. \quad (4.2a, b)$$

Combining the sets S_1 and S_2 by using Eqs. (4.1) and (4.2a,b), we obtain $\Omega_{N,\phi}$ as

$$\Omega_{N,\phi} = S_1 \cup S_2 = \{r, b, r + b, 2b, \dots, r + (p - 1)b, pb\}. \quad (4.3)$$

Comparing (4.3) with (2.20), and also using (2.21), it is easy to see that the sizes of the clusters are given by

$$\{\{r, b - r, r, b - r, \dots, r, b - r, r\}\}. \quad (4.4)$$

Hence, for any fraction of type I, the corresponding Bethe state contains $(p + 1)$ number of clusters of size r and p number of clusters of size $(b - r)$. Next, by using the method of contradiction, we would like to show that these two possible sizes of the clusters, i.e., r and $(b - r)$, must be relatively prime integers. To this end, let us first assume that b and r are not relatively prime integers. Therefore, we can write b and r as $b = \alpha b'$ and $r = \alpha r'$, where $\alpha > 1$. Substituting these values of b and r in Eq. (4.1), we find that

$$N = \alpha(pb' + r').$$

Thus α is a common factor of N and b . However, this result contradicts the fact that $\{N, b\}$ must be relatively prime integers for any fraction of type I. Hence it is established that $\{b, r\}$ are relatively prime integers. From this relation, it trivially follows that $\{r, b - r\}$ are relatively prime integers and, in particular, $r \neq b - r$. Consequently from Eq. (4.4) we find that, for any fraction of type I, the corresponding Bethe state contains heterogeneous clusters of two different sizes. As a special case, let us consider any

fraction of type I with denominator satisfying the relation $b > N/2$. Due to Eq. (4.1) it follows that, $p = 1$, $r = N - b$ and $b - r = 2b - N$ for this case. Hence, the corresponding Bethe state contains two clusters of the size $(N - b)$ and one cluster of the size $(2b - N)$.

Next, we consider the clusters of bound particles corresponding to any fraction of type III. In this case N can be written as $N = pb$, where p is an integer greater than one. Consequently, for the variable l taking value within the set Ω_N , the zero points of the functions $\sin l\phi$ and $\sin(N - l)\phi$ coincide with each other and yield $\Omega_{N,\phi}$ as

$$\Omega_{N,\phi} = \{b, 2b, 3b, \dots, (p - 1)b\}. \quad (4.5)$$

Hence p number of clusters are formed in this case. Comparing (4.5) with (2.20), and also using (2.21), it is easy to see that each cluster of this type has the size b . In other words, the corresponding Bethe state (2.3) contains N/b of homogeneous clusters, each of which is made of b number of bound particles. In Table 1 we show all the fractional values of ϕ/π for which clusters of bound particles exist within the range of N given by $4 \leq N \leq 10$, the types of these fractions and the sizes of the corresponding clusters using the notation of Eq. (2.21).

4.2 Stability of clusters of bound particles and binding energies

Here, we shall discuss about the stability of clusters of bound particles under infinitesimal variations of the coupling constant. For $N \geq 4$, let us choose any specific value of the coupling constant ϕ within the range $0 < \phi < \frac{\pi}{2}$ such that clusters of bound particles are formed. If one increases or decreases this value of ϕ by an infinitesimal amount, it is obvious that all inequalities in Eq. (3.1b) would continue to be satisfied and all equalities in Eq. (3.1a) would be transformed into some inequalities. Consequently, clusters of bound particles cease to exist even for a very small change of the coupling constant. One of the following two different cases can occur in such a situation. In the first case, at least one of the equalities in Eq. (3.1a) is transformed into an inequality of the form

$$\chi \sin(l\phi) \sin[(N - l)\phi] < 0. \quad (4.6)$$

It is evident that, the probability density of the corresponding Bethe state (2.3) would diverge if the relative distance between at least one pair of particle coordinates tends towards infinity. As a result, this Bethe state becomes ill-defined and disappears from the Hilbert space of the Hamiltonian (1.1) of derivative the δ -function Bose gas. So we may say that clusters of bound particles become unstable in this case. Let us now consider the second case, for which all of equalities in Eq. (3.1a) are transformed into inequalities of the form

$$\chi \sin(l\phi) \sin[(N - l)\phi] > 0, \quad (4.7)$$

due to an infinitesimal change of the coupling constant ϕ . It is evident that, for this case, Eqs. (3.1a,b) are transformed to Eq. (2.16) within the range of ϕ given by $0 < \phi < \frac{\pi}{2}$. As a result, clusters of bound particles merge with each other and produce a localized

N	Value of ϕ/π	Type	Size of the clusters
4	1/3	I	$\{\{1, 2, 1\}\}$
5	1/4	I	$\{\{1, 3, 1\}\}$
5	1/3	I	$\{\{2, 1, 2\}\}$
6	1/5	I	$\{\{1, 4, 1\}\}$
6	1/3	III	$\{\{3, 3\}\}$
7	1/6	I	$\{\{1, 5, 1\}\}$
7	1/4	I	$\{\{3, 1, 3\}\}$
7	1/3	I	$\{\{1, 2, 1, 2, 1\}\}$
7	2/5	I	$\{\{2, 3, 2\}\}$
8	1/7	I	$\{\{1, 6, 1\}\}$
8	1/4	III	$\{\{4, 4\}\}$
8	1/3	I	$\{\{2, 1, 2, 1, 2\}\}$
8	2/5	I	$\{\{3, 2, 3\}\}$
9	1/8	I	$\{\{1, 7, 1\}\}$
9	1/5	I	$\{\{4, 1, 4\}\}$
9	1/4	I	$\{\{1, 3, 1, 3, 1\}\}$
9	1/3	III	$\{\{3, 3, 3\}\}$
9	3/7	I	$\{\{2, 5, 2\}\}$
10	1/9	I	$\{\{1, 8, 1\}\}$
10	1/5	III	$\{\{5, 5\}\}$
10	2/7	I	$\{\{3, 4, 3\}\}$
10	1/3	I	$\{\{1, 2, 1, 2, 1, 2, 1\}\}$
10	2/5	III	$\{\{5, 5\}\}$

Table 1: The fractional values of ϕ/π for which clusters of bound particles exist for $4 \leq N \leq 10$, the types of these fractions and the sizes of the corresponding clusters are shown.

bound state containing only one cluster of particles. Therefore, in this second case, we may say that clusters of bound particles turn into a localized bound state with only one cluster.

Using the above mentioned procedure, we have found that [43] clusters of bound particles associated with fractions of type III become unstable and cease to exist for any small change of the coupling constant. On the other hand, for the case of fractions of type I, clusters of bound particles transmute to a localized bound state containing only one cluster if the value of ϕ is slightly changed towards the direction of the nearest fraction n/N . However, such clusters of bound particles become unstable if the value of ϕ is slightly changed towards the opposite direction. Consequently, fractions of type I lie at the end points of the bands containing localized bound states.

5 Conclusion

In this article, we have explored how clusters of bound particles can be constructed in the simplest possible way for the case of an exactly solvable derivative δ -function Bose gas. To this end, we consider a sufficient condition for which Bethe states of the form in (2.3) would lead to clusters of bound particles. It is found that this sufficient condition can be satisfied by taking the quasi-momenta of the corresponding Bethe state to be equidistant points on a *single* circle having its centre at the origin of the complex momentum plane. Furthermore, the coupling constant (ϕ) and the total number of particles (N) of this derivative δ -function Bose gas must satisfy the relations in (3.1a,b). For any given $N \geq 4$, it is found that Eq. (3.1a) is satisfied if ϕ/π takes any value within the set F'_{N-1} , which is a subset of the Farey sequence F_{N-1} . Then we classify all fractions belonging to the set F'_{N-1} into four types. It turns out that fractions of types I and III belonging to F'_{N-1} satisfy the remaining relation (3.1b), while type II and type IV fractions do not satisfy (3.1b). Consequently, clusters of bound particles can be constructed for the derivative δ -function Bose gas only for special values of ϕ/π given by the fractions of types I and III within the set F'_{N-1} .

We have also computed the sizes of the above mentioned clusters of bound particles, i.e., the number of particles present within each of these clusters. It is found that any fraction of type I within the set F'_{N-1} leads to heterogeneous clusters of bound particles having two different sizes. On the other hand, any fraction of type III within the set F'_{N-1} leads to homogeneous clusters of bound particles having only one size. Interestingly, clusters of bound particles associated with fractions of type I and type III transform in rather different ways under a small variation of the coupling constant. For example, it is found that clusters of bound particles associated with fractions of type III cease to exist for any small change of the coupling constant. On the other hand, clusters of bound particles corresponding to fractions of type I turn into localized bound states consisting of a single cluster if the value of the coupling constant is slightly increased or decreased.

In this paper we have analyzed a particular type of sufficient condition for constructing clusters of bound particles in the case of the derivative δ -function Bose gas in the finite volume limit. However, in future, it might be interesting to explore other possible ways of constructing clusters of bound particles for this exactly solvable system. As is well known, many properties of the complex quasi-momenta for the case of the δ -function Bose gas can be derived relatively easily by using the solutions of the corresponding finite volume Bethe ansatz equations. A similar analysis of the finite volume Bethe ansatz equations may be helpful in finding all possible clusters of bound particles for the derivative δ -function Bose gas in the infinite volume limit.

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